ICCOMSET-2019

ABS-217

Logged in as Admin
» Logout

» Home
» Software License
» Admin Guide
» FAQs
» Data Summary
» Users
» New Users
» Statistics
» Abstracts
» Statistics
» Data Table
» Data Checking
» Payment Proofs
» Payment Invoice
» Full Papers
» Paper Review
» Statistics
» Data Table
» Manage
» Result/Action
» Archive
» Revised Papers
» Copyright Transfers
» Presentation Video
» Online Q&A Forum
» Certificate
» Presenter
» Non-Presenter
» Time Series Graphs
» File Data Table
» Email List

:: User profile ::

<< back

Name	Dr. Aceng Sambas [Profiling | GS Record | Delete User]
User ID	USER-327 (Activated on Monday, 20 July 2020)
Email	acengs@umtas.ac.id
Login Code	xsZ9BTYban
Institution	Universitas Muhammadiyah Tasikmalaya
Research	Dynamical System
Participation	Presenter
Postal address	Street: Tamansari Gobras Km. 2.5; City: Tasikmalaya; ZIP code: 46196; Country: Indonesia
Phone number	
Fax number	
Other information	aqua
Status	Approved

There is no guarantee that your request will be accommodated

[Abstract ID: ABS-217]
Search on Ifory

Stability of Lorenz System at the Second Equilibria Point based on Gardano^s Method

S. F. AL-Azzawi1,, Mujiarto2 , L. Patria3 , A. Sambas2 and W. S. M. Sanjaya4*

1 Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq
2 Department of Mechanical Engineering, Universitas Muhammadiyah Tasikmalaya, Indonesia
3 Department of Information Systems, Universitas Terbuka, Indonesia
4 Department of Physics, Universitas Islam Negeri Sunan Gunung Djati, Bandung, Indonesia

Abstract

In this paper, stability conditions of the Lorenz system at the second equilibrium point are investigated by applying Gardano^s method where the system has three equilibria points. Most of the previous work focused their studies at the original point. A few studies demonstrated stability of dynamical systems at another equilibria points by use of conventional techniques. However, it is often unclear and based on numerical methods. This reason, motivate us to establish the stability conditions of the Lorenz system at a point which different from the origin point and compare between them. Finally, An illustrative example shows the effectiveness and feasibility of this method. **(Approx. 104 words)**

Keywords: Lorenz system, Gardanos method

Topic: Engineering and Technology

Type: Oral Presentation

Info:

Abstract Review Result

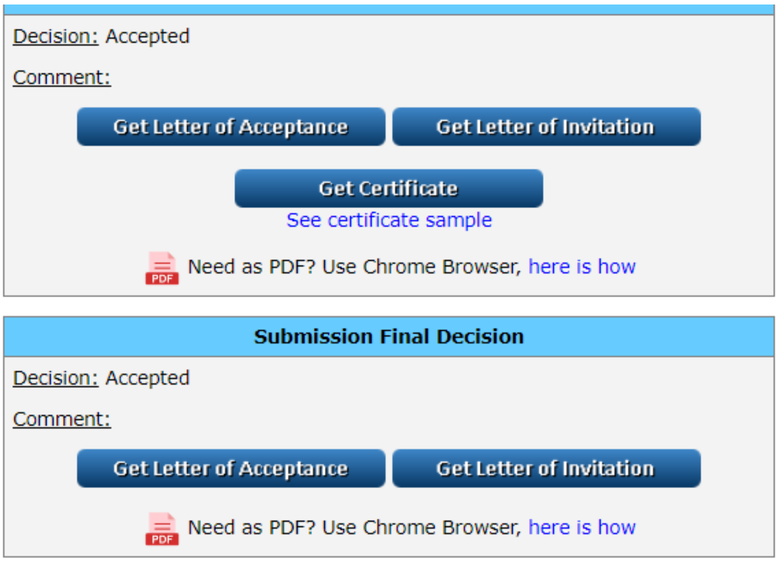
Decision: Accepted

Comment:

Get Letter of Acceptance | Get Letter of Invitation

Get Certificate
See certificate sample

Need as PDF? Use Chrome Browser, here is how



Decision: Accepted

Comment:

Get Letter of Acceptance | Get Letter of Invitation

Get Certificate

See certificate sample

Need as PDF? Use Chrome Browser, here is how

Submission Final Decision

Decision: Accepted

Comment:

Get Letter of Acceptance | Get Letter of Invitation

Need as PDF? Use Chrome Browser, here is how

URL JPCS-1477: https://iopscience.iop.org/volume/1742-6596/1477

URL pdf: https://iopscience.iop.org/article/10.1088/1742-6596/1477/2/022009/pdf

URl abstract: https://iopscience.iop.org/article/10.1088/1742-6596/1477/2/022009

Link indexing: https://www.scimagojr.com/journalsearch.php?q=130053&tip=sid&clean=0

Print this page

ICComSET 2019

The 2nd International Conference on Computer, Science, Engineeringand Technology
Universitas Muhammadiyah Tasikmalaya/Banten, 15-16 October 2019
Website: http://www.2ndiccomset.umtas.ac.id
Email: iccomset@umtas.ac.id

Date: 08 July 2019

Letter of Acceptance for Abstract

Dear Authors: S. F. AL-Azzawi1,*, Mujiarto2 , L. Patria3 , A. Sambas2 and W. S. M. Sanjaya4

We are pleased to inform you that your abstract (ABS-217, Oral Presentation), entitled:

"Stability of Lorenz System at the Second Equilibria Point based on Gardano^s Method"

has been reviewed and accepted to be presented at ICComSET 2019 conference to be held on
15-16 October 2019, in Banten, Indonesia.

Please submit your full paper and make the payment for registration fee before the deadlines, visit our website for more information.

Thank You.

Best regards,

Dr. Mujiarto, S.T.,M.T.
ICComSET 2019Chairperson

Print this page

ICComSET 2019

The 2nd International Conference on Computer, Science, Engineeringand Technology
Universitas Muhammadiyah Tasikmalaya/Banten, 15-16 October 2019
Website: http://www.2ndiccomset.umtas.ac.id
Email: iccomset@umtas.ac.id

Date: 11 October 2019

Letter of Acceptance for Full Paper

Dear Authors: S. F. AL-Azzawi1,*, Mujiarto2 , L. Patria3 , A. Sambas2 and W. S. M. Sanjaya4

We are pleased to inform you that your paper, entitled:

"Stability of Lorenz System at the Second Equilibria Point based on Gardano^s Method"

has been reviewed and accepted to be presented at ICComSET 2019 conference to be held on 15-16 October 2019 in Banten, Indonesia.

Please make the payment for registration fee before the deadlines, visit our website for more information.

Thank You.

Best regards,

Dr. Mujiarto, S.T.,M.T.
ICComSET 2019 Chairperson

Print this page

ICComSET 2019

The 2nd International Conference on Computer, Science, Engineeringand Technology
Universitas Muhammadiyah Tasikmalaya/Banten, 15-16 October 2019
Website: http://www.2ndiccomset.umtas.ac.id
Email: iccomset@umtas.ac.id

Date: 10 September 2019

Letter of Invitation

Dear Authors: S. F. AL-Azzawi1,*, Mujiarto2 , L. Patria3 , A. Sambas2 and W. S. M. Sanjaya4

We are pleased to inform you that your abstract (ABS-217, Oral Presentation), entitled:

"Stability of Lorenz System at the Second Equilibria Point based on Gardano^s Method"

has been reviewed and accepted to be presented at ICComSET 2019 conference to be held on15-16 October 2019 in Banten, Indonesia.

We cordially invite you to attend our conference and present your research described in the abstract.

Please submit your full paper and make the payment for registration fee before the deadlines, visit our website for more information.

Thank You.

Best regards,

Dr. Mujiarto, S.T.,M.T.
ICComSET 2019 Chairperson

Print this page

ICComSET 2019

The 2ndInternational Conference on Computer, Science, Engineering and Technology
Universitas Muhammadiyah Tasikmalaya/Banten, 15-16 October 2019
Website:http://www.2ndiccomset.umtas.ac.id
Email: iccomset@umtas.ac.id

Date: 10 September 2019

Letter of Invitation

Dear Authors: S. F. AL-Azzawi1,*, Mujiarto2 , L. Patria3 , A. Sambas2 and W. S. M. Sanjaya4

We are pleased to inform you that your paper, entitled:

"Stability of Lorenz System at the Second Equilibria Point based on Gardano^s Method"

has been reviewed and accepted to be presented at ICComSET 2020 conference to be held on
15-16 October 2019 in Banten, Indonesia.

We cordially invite you to attend our conference and present your research described in the paper.

Please make the payment for registration fee before the deadlines, visit our website for more information.

Thank You.

Best regards,

Dr. Mujiarto, S.T.,M.T.
ICComSET 2019 Chairperson

Stability of Lorenz System at the Second Equilibria Point based on Gardano's Method

S. F. AL-Azzawi[1,*], Mujiarto[2], L. Patria[3], A. Sambas[2] and W. S. M. Sanjaya[4]

[1] Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq

[2] Department of Mechanical Engineering, Universitas Muhammadiyah Tasikmalaya, Indonesia

[3] Department of Information Systems, Universitas Terbuka, Indonesia

[4] Department of Physics, Universitas Islam Negeri Sunan Gunung Djati, Bandung, Indonesia

*saad_fawzi78@yahoo.com

Abstract. In this paper, stability conditions of the Lorenz system at the second equilibrium point are investigated by applying Gardano's method where the system has three equilibria points. Most of the previous work focused their studies at the original point. A few studies demonstrated stability of dynamical systems at another equilibria points by use of conventional techniques. However, it is often unclear and based on numerical methods. This reason, motivate us to establish the stability conditions of the Lorenz system at a point which different from the origin point and compare between them. Finally, An illustrative example shows the effectiveness and feasibility of this method.

1. Introduction

In 1963, Lorenz found the first chaotic system, which is a third order autonomous system with only two multiplication-type quadratic terms, but displays very complex dynamical behaviors [1,2]. By definition Vanecek and Celikovsky the Lorenz system satisfies the condition $a_{12}a_{21} > 0$, where a_{12} and a_{21} are corresponding elements in the constant matrix $A = (a_{ij})_{3\times 3}$ for the linear part of system [3].

Lorenz system is not integrable and it is difficult to find an analytical solution for this system in three dimension parameters space, but special cases for Lorenz system are studied before studying periodic solutions, and Lorenz studied the system when $\sigma = 10$, $\beta = 8/3$. From the definition of equilibrium points, it is easy to verify that, when $r \leq 1$ the Lorenz system has only one equilibrium point, which is the origin, but when $r > 1$ it has three equilibria points: $E_1(0,0,0)$ $E_{2,3}(\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}\ , r-1)$ [4,5], The Lorenz system has some simple properties such that this system has natural symmetry (x,y,z)$\rightarrow$(-x,-y,z) and the z-axis is invariant [6,7].

[8,9] discussed the stability of Lorenz system about the equilibria points, and found the roots of characteristic equation for this system at origin, but at the second equilibrium point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}\ , r-1)$, the Ref [3] used Routh-Hurwitz test to investigated the stability without founding the roots, the Routh-Hurwitz paly important role in stability of dynamical systems

[10], while the Ref [5] depended on the value of r to investigated the stability without founding the roots.

In [9] studied the stability for system derived from the Lorenz system and depended on the roots to determine the stability at origin, The determination of the roots of a cubic equation in general is fairly difficult, but in the given case, one root is easily found [11]. We can find the roots of equations for third degree by numerical method, and these roots are proximal (not exact), but by using Gardano's method on the same equations we can find exact roots, and by these roots we can investigated the stability for any system.

In this paper, the stability conditions of Lorenz system at the second equilibrium point E_2 is established by using the general formula Gardano's method to find the roots of the characteristic equation for this system. The Lorenz system is described by:

$$\begin{cases} x = \sigma(y - x) \\ y = rx - y - xz \\ z = xy - \beta z \end{cases} \tag{1}$$

Where σ, r, β are positive parameters. Figure 1 and Figure 2 shows the attractors of the system (1).

The approximating linear system (1) at E_1 is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2}$$

or the characteristic equation of the form:

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{3}$$

Then

$$\begin{cases} a = \sigma + \beta + 1 \\ b = \beta(\sigma + 1) + \sigma(1 - r) \\ c = \beta\sigma(1 - r) \end{cases} \tag{4}$$

The solutions of Eq. 3 are

$$\lambda_{1,2} = \frac{1}{2}\left[-\sigma - 1 \pm \sqrt{(\sigma - 1)^2 + 4\sigma r}\ \right], \quad \lambda_3 = -\beta \tag{5}$$

Now consider the system (1) at second equilibrium point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$, Under the linear transformations $(x, y, z) \to (X, Y, Z)$,

$$\begin{cases} x = X + \sqrt{\beta(r-1)} \\ y = Y + \sqrt{\beta(r-1)} \\ z = Z + (r-1) \end{cases} \tag{6}$$

The system (1) becomes

$$\begin{cases} X = -\sigma X + \sigma Y \\ Y = X - Y - \sqrt{\beta(r-1)}\, Z \\ Z = \sqrt{\beta(r-1)} X + \sqrt{\beta(r-1)}\, Y - \beta Z \end{cases} \tag{7}$$

The approximating linear system (1) at equilibrium point E_2 is:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(r-1)} \\ \sqrt{\beta(r-1)} & \sqrt{\beta(r-1)} & -\beta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{8}$$

And the characteristic equation of the form:

$$\lambda^3 + a_1\lambda^2 + b_1\lambda + c_1 = 0 \quad (9)$$

Then

$$\begin{cases} a = a_1 = \sigma + \beta + 1 \\ b_1 = \beta(\sigma + r) \\ c_1 = 2\beta\sigma(r - 1) \end{cases} \quad (10)$$

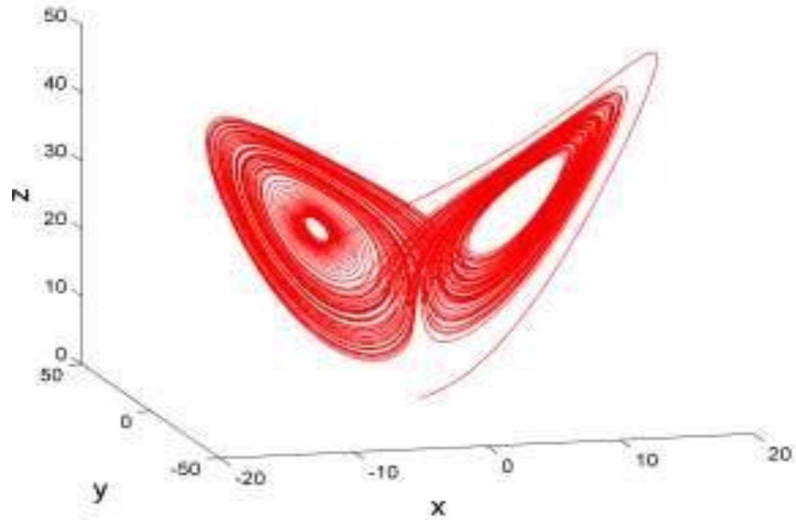


Figure 1 The attractor of system (1) in x, y, z space.

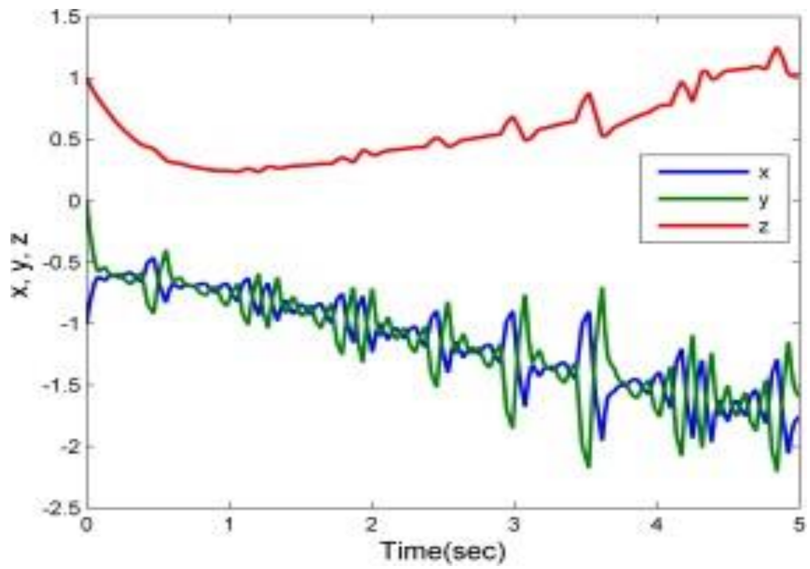


Figure 2 The attractor of system (1).

2. Helping results

Remark 1[5]:

When $\sigma = 10, \beta = 8/3$, the solutions of equation (9) depend on the parameter r as follows:

1- For $1 < r < r_1 \cong 1.3456$, there are three negative real roots,

2- For $r_1 < r < r_2 \cong 24.737$, there are one negative real root and two complex roots with negative real parts,

3- For $r > r_2$ there are one negative real root and two complex roots with positive real parts.

Remark 2[4]:

Let A be a $n \times n$ matrix of constants. A equilibrium point for the system $\mathrm{X} = Ax$ is

- aymptotically stable if all roots of A has negative real parts
- unstable if A has at least one root with a positive real parts.

Remark 3[10]: **Critical case**

In critical cases when the real parts of all roots of the characteristic equation are non positive, with the real part of at least one root being zero.

Remark 4[3]: Critical value

Lorenz system has critical value which is $r_c = 1$ at origin and $r_c = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}$ at the second equilibrium point, and this system is asymptotically stable if r lies between 1 and a critical value r_c at $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$.

Let us denote

$$q = c_1 - \frac{1}{3} a_1 b_1 + \frac{2}{27} a_1^3 \tag{11}$$

$$\Delta = c_1^2 + \frac{4}{27} b_1^3 - \frac{2}{3} a_1 b_1 c_1 - \frac{1}{27} a_1^2 b_1^2 + \frac{4}{27} a_1^3 c_1 \tag{12}$$

We will use the following theorem, which enables us to find the exact roots for cubic equation (three degree).

Theorem 1[12, 13]: (Gardano's method)

- If $\Delta = 0$, then the second term of equation (9) has three roots, but one is multiple:

$$\lambda_1 = -2\sqrt[3]{\frac{q}{2}} - \frac{a1}{3}, \quad \lambda_{2,3} = \sqrt[3]{\frac{q}{2}} - \frac{a1}{3} \tag{13}$$

- If $\Delta < 0$, then the equation (10) has three different real roots as:

$$\lambda_{i+1} = \sqrt[6]{16(q^2 - \Delta)} \cos \frac{\cos^{-1}\frac{-g}{\sqrt{g^2-\Delta}} + 2\pi i}{3} - \frac{a1}{3}, \qquad i = 0,1,2. \tag{14}$$

- If $\Delta > 0$, then the equation (10) has one real root and two complexes conjugate roots with non-vanishing imaginary parts as:

$$\begin{cases} \lambda_1 = \sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}} - \frac{a_1}{3} \\ \lambda_2 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a_1}{3} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) \\ \lambda_3 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a_1}{3} - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) \end{cases} \tag{15}$$

3. Main results:

Corollary 1: If $\sigma = 10, \beta = 8/3$ and

- $r = r_1$, then we have three negative real roots.
- $r_1 = r_2$, then we have one negative real root and two complex roots with vanishing real parts.

Theorem 2: The solutions of system (1) at second point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$ when $\sigma = 10, \beta = 8/3$ are:

- Asymptotically stable if the following cases hold:
  (i) $\Delta < 0$ and $r \in (1, r_1]$
  (ii) $\Delta > 0$ and $r \in (r_1, r_2)$
- Unstable if $\Delta > 0$ and $r \in (r_2, \infty)$
- Critical case if $\Delta > 0$ and $r = r_2$

Proof:
Case 1: By theorem (1) when $\Delta < 0$ we obtain:
$\lambda_1, \lambda_2, \lambda_3$ are different real roots and these roots are negative when $1 < r \leq r_1$, (by Remark 1.1 and corollary1.1), hence satisfied Remark 2.1, therefore the system (1) is asymptotically stable,

When $\Delta > 0$ we obtain:
λ_1 is a real root and λ_2, λ_3 are complex conjugate roots and these roots are negative (negative real parts) when and $r \in (r_1, r_2)$ and satisfied remark 2.1, therefore the system (1) is asymptotically stable.
Case 2: when $\Delta > 0$, we have λ_1 is a real root and λ_2, λ_3 are complex conjugate roots and by remark 1.3.When $r \in (r_2, \infty)$, we obtain that λ_1 is negative and λ_2, λ_3 are positive real part, hence satisfied remark 2,2 , therefore the system (1) is unstable.
Case 3: When $r = r_2$ we have λ_1 is negative real root and $\text{Re } \lambda_2 = \text{Re } \lambda_3 = 0$ (Corollary1.2) and satisfied remark 3, hence the system (1) is a critical case, the proof is complete.

We can generalization Theorem (2) for any value of σ and β in the following theorem
Theorem 3: The solutions of system (1) at second critical point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$ are:

- Asymptotically stable if the following cases hold:
  (i) $\Delta < 0$, $\lambda_2, \lambda_3 < 0$
  (ii) $\Delta > 0$, $\text{Re } \lambda_2 < 0$
- Unstable if $\Delta > 0$, $\text{Re } \lambda_2 > 0$

Critical case if $\Delta > 0$, $\text{Re } \lambda_2 > 0$

Proof:
Case 1. By theorem (1) when $\Delta < 0$ we obtain: a three different real roots and λ_2, λ_3 are negative (given), we must prove that λ_1 is negative, Since the cubic equation (10) has a positive coefficients therefore, then at least one of these roots is negative real part, hence satisfied remark 2,1 and the system (1) is asymptotically stable,

When $\Delta > 0$, then we have $\text{Re } \lambda_3 < 0$ also since $\text{Re } \lambda_2 = \text{Re } \lambda_3$ (two complex conjugate roots) and λ_1 is negative real root (cubic equation with positive coefficients must have a negative real root), then we have one negative real root and complex roots with negative real part, hence the system (1) is asymptotically stable.

Case 2. By the same theorem when $\Delta > 0$, we have Re $\lambda_3 > 0$ since Re $\lambda_2 =$ Re λ_3 (analogously as in proof of case 1) and λ_1 is negative real root, then we have one negative real root and two complex roots with positive real part, hence satisfied remark 2,2 , the system (1) is unstable.
Case 3. when $\Delta > 0$ and Re $\lambda_2 = 0$, then Re $\lambda_3 = 0$ and λ_1 is negative real part, hence satisfied remark 3, the system (1) is a critical case, the proof is complete.

Proposition:

The necessary and sufficient condition for a second critical point whose parameter σ is greater than the parameter β plus one .

Proof:

1- If $\sigma = \beta$ then critical value become $r_c = -\sigma(\sigma + \beta + 3)$, (contradictions) since r_c must larger then 1.
2- If $\sigma < \beta$ then denominator of critical value is always negative therefore r_c is a negative (contradictions) the same reason of the first case.
3- If $\sigma > \beta$, then critical value is positive and larger then 1(possible), but if $\sigma = \beta + 1$ then critical value become $r_c = \frac{\sigma(\sigma+\beta+3)}{0}$, (contradictions) not allowed dividing on a zero, therefore we must $\sigma > \beta + 1$ only.

4. Comparison

The following Table 1 distinguish the most important differences between the first critical point $E_1(0,0,0)$ and the second point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$.

Table 1 Comparison between equilibria points: E_1, E_2

Nu.	At $E_1(0,0,0)$	**Nu.**	$E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$,
1-	r is any number larger then zero	1-	r is any number larger then one
2-	A critical value is fixed	2-	A critical value is not fixed, changes with the initial data σ, β
3-	It is easy to determination one root $\lambda = -\beta$	3-	It is difficult to determine one root
4-	Contain only real roots	4-	Contain real and complex roots
5-	Not all coefficients of characteristics equation are positive	5-	all coefficients of characteristics equation are positive
6-	Asymptotically stable when $r \in (0,1)$	6-	Asymptotically stable when $r \in (1, r_c)$
7-	Unstable when $r \in (1, \infty)$	7-	Unstable when $r \in (r_c, \infty)$
8-	Critical case when $r = 1$	8-	Critical case when $r = r_c$
9-	$\sigma = \beta$, $\sigma < \beta$, $\sigma > \beta$	9-	$\sigma > \beta + 1$ only
10 -	q may by negative or positive or zero	10 -	q is a positive only
11-	Δ may by negative or zero	11-	Δ may by negative or positive
12 -	Contain on multiple real roots	12 -	Not contain on multiple real roots

Figures 3, 4 show the stability at equilibria points E_1, E_2 respectively.

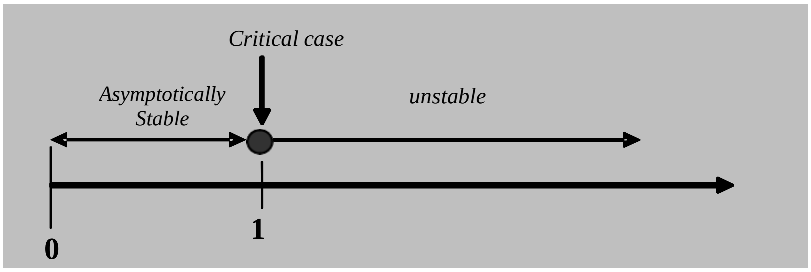


Figure 3 Stability at $E_1(0,0,0)$

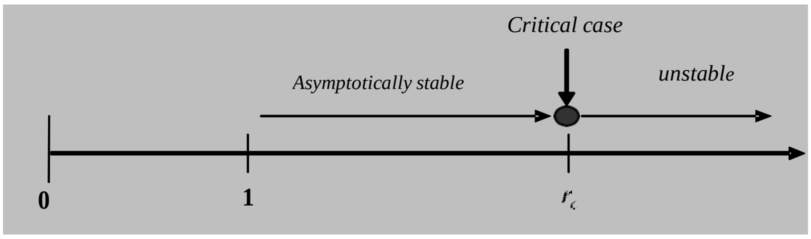


Figure 4 Stability at $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$

5. Illustrative Examples:

In this section, we take two different systems, for example to show how to use the results obtained in this paper to analyze the stability of class chaotic systems.

Example 1: Investigate for stability of the following Lorenz system

$$\begin{cases} x = -10x + 10y \\ y = 11x - y - xz \\ z = xy - \frac{8}{3}z \end{cases}$$

The characteristic equation of Lorenz system is of the form: $\lambda^3 + \frac{41}{3}\lambda^2 + 56\lambda + \frac{1600}{3} = 0$

$q = \frac{22898}{49}$, $\Delta = \frac{436677}{2} > 0$, $11 \in (r_1, r_2)$ and $\lambda_1 = -\frac{1807}{150}$, $\lambda_{2,3} = -\frac{1961}{2421} \pm \frac{1634}{235}\mathrm{i}$

Then the system (1) is asymptotically stable

<u>In case r = 100</u>

The characteristic equation of Lorenz system is of the form: $\lambda^3 + \frac{41}{3}\lambda^2 + \frac{880}{3}\lambda + 5280 = 0$

$q = \frac{57859}{14}$, $\Delta = 18907821 > 0$, $100 \in (r_2, \infty)$ and $\lambda_1 = -\frac{8407}{526}$, $\lambda_{2,3} = \frac{564}{487} \pm \frac{2485}{137}\mathrm{i}$

Then the system (1) is unstable. Figure 5 and figure 6 show the attractors of system(1) when $\sigma = 10, \beta = 8/3$, (a) $r = 11$, (b) $r = 100$.

Example 2: Investigate for stability of the following Lorenz system

$$\begin{cases} x = -7x + 7y \\ y = \frac{3}{2}x - y - xz \\ z = xy - 4z \end{cases}$$

The characteristic equation of Lorenz system is of the form: $\lambda^3 + 12\lambda^2 + 34\lambda + 28 = 0$

$q = 20$, $\Delta = -\frac{176}{27} < 0$, $\frac{3}{2} \in (1, r_c)$ and $\lambda_1 = -2$, $\lambda_2 = -\frac{2015}{1197}$ $\lambda_3 = -\frac{3152}{379}$

Then the system (1) is asymptotically stable

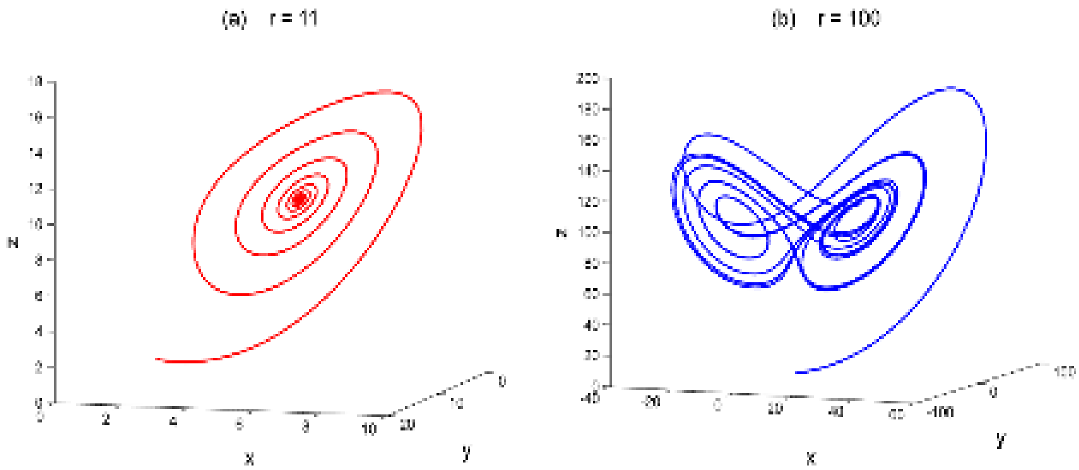


Figure 5 The attractor of system(1) when $\sigma = 10, \beta = 8/3$ (a) $r = 11$, (b) $r = 100$

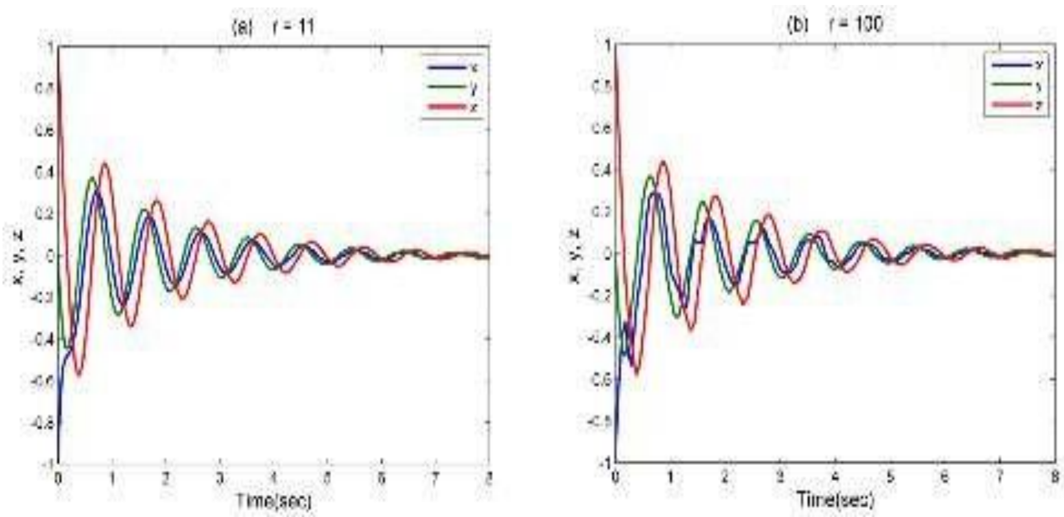


Figure 6 The attractor of system(1) convergent to zero when $\sigma = 10, \beta = 8/3$
(a) $r = 11$, (b) $r = 100$

<u>In case r = 49</u>

The characteristic equation of Lorenz system is of the form: $\lambda^3 + 12\lambda^2 + 224\lambda + 2688 = 0$

$q = 1920$, $\Delta = 4494071 > 0$, $49 = r_c$ and $\lambda_1 = -12$, $\lambda_{2,3} = \pm\frac{13455}{899}i$

Then the system (1) is a critical case. Figure 7 and figure 8 show the attractors of system(1) when $\sigma = 7, \beta = 4$, (a) $r = 3/2$, (b) $r = 49$.

6. Conclusions

In this paper, we have investigated the stability of Lorenz system at the second critical point by using a new method. By this method we justified the same results which found by previous methods. An illustrative examples show the effectiveness and feasibility of this method.

References

[1] Li D, Lu J A, Wu X and Chen G 2005 *Chaos, Solitons & Fractals* **23** 529-534

[2] Al-Obeidi A S and AL-Azzawi S F 2019 *Indonesian Journal of Electrical Engineering and Computer Science* **16** 692-700

[3] Al-Azzawi S F 2012 *Applied Mathematics and Computation* **219** 1144-1152

[4] Borrelli R L and Coleman C S 1998 *Differential Equations* (New York: John Wiley and Sons)

[5] Boyce W E and Diprima R C 2004 *Elementary Differential Equations and Boundary Value Problems* (New York: John Wiley and Sons, Inc)

[6] Zhang, Zeng Y and Li Z 2018 *Chinese Journal of Physics* **56** 793-806

[7] Sambas A, Vaidyanathan S, Zhang S, Zeng Y, Mohamed M A and Mamat M 2019 *IEEE Access* **7** 115454-115462

[8] Lynch S 2001 *Dynamical Systems with Applications using Maple* (Basel: Birkhauser)

[9] Tigan G 2004 *Proceedings of the Third International Colloquium on Mathematics in Engineering and Numerical Physics* 265-272

[10] AL-Azzawi S F and Aziz M M 2019 *Telkomnika* **17** 1931-1940

[11] Ambrosio L and Dancer N 2012 *Calculus of variations and partial differential equations: topics on geometrical evolution problems and degree theory* (German: Springer Science & Business Media)

[12] Aziz M M and Al-Azzawi S F 2017 *Optik* **138** 328-340

[13] Al-Azzawi S F and Aziz M M 2018 *Alexandria engineering journal* **57** 3493-3500

Print this page

ICComSET 2019

The 2nd International Conference on Computer, Science, Engineeringand Technology
Universitas Muhammadiyah Tasikmalaya/Banten, 15-16 October 2019
Website: http://www.2ndiccomset.umtas.ac.id
Email: iccomset@umtas.ac.id

Date: 08 October 2019

Payment Invoice

Submission Title	Stability of Lorenz System at the Second Equilibria Point based on Gardano^s Method
Authors	S. F. AL-Azzawi1,*, Mujiarto2 , L. Patria3 , A. Sambas2 and W. S. M. Sanjaya4
Registration Type	Indonesian Presenter
Payment Amount	IDR 2,850,000 (**Not Paid**)

Payment Account	
Bank Name	Bank Syariah Mandiri
Account Number	7142050476
Account Holder	LPPM UMTAS
Info	BSMDIDJAXXX

Note that this document is NOT receipt of payment, please make the payment and then upload your payment proof to the online system.

Best regards,

Anggia Suci Pratiwi, M.Pd.
ICComSET 2019 Finance Manager

Print this page

ICComSET 2019

The 2nd International Conference on Computer, Science, Engineeringand Technology
Universitas Muhammadiyah Tasikmalaya/Banten, 15-16 October 2019
Website: http://www.2ndiccomset.umtas.ac.id
Email: iccomset@umtas.ac.id

Date: 11 October 2019

Payment Receipt

The organizing committee of ICComSET 2020 acknowledges the following payment for registration fee,

Abstract ID	ABS-217 (Oral Presentation)
Title	"Stability of Lorenz System at the Second Equilibria Point based on Gardano^s Method"
Authors	S. F. AL-Azzawi1,*, Mujiarto2 , L. Patria3 , A. Sambas2 and W. S. M. Sanjaya4
Paid Amount	IDR 2,850,000
Paid By	Mr. A. Sambas

Thank You.

Best regards,

Anggia Suci Pratiwi, M.Pd.
ICComSET 2019 Finance Manager

Stability of Lorenz System at the Second Equilibria Point based on Gardano's Method

S. F. AL-Azzawi[1,*], Mujiarto[2], L. Patria[3], A. Sambas[2] and W. S. M. Sanjaya[4]

[1] Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq
[2] Department of Mechanical Engineering, Universitas Muhammadiyah Tasikmalaya, Indonesia
[3] Department of Information Systems, Universitas Terbuka, Indonesia
[4] Department of Physics, Universitas Islam Negeri Sunan Gunung Djati, Bandung, Indonesia

*saad_fawzi78@yahoo.com

Abstract. In this paper, stability conditions of the Lorenz system at the second equilibrium point are investigated by applying Gardano's method where the system has three equilibria points. Most of the previous work focused their studies at the original point. A few studies demonstrated stability of dynamical systems at another equilibria points by use of conventional techniques. However, it is often unclear and based on numerical methods. This reason, motivate us to establish the stability conditions of the Lorenz system at a point which different from the origin point and compare between them. Finally, An illustrative example shows the effectiveness and feasibility of this method.

1. Introduction

In 1963, Lorenz found the first chaotic system, which is a third order autonomous system with only two multiplication-type quadratic terms, but displays very complex dynamical behaviors [1,2]. By definition Vanecek and Celikovsky the Lorenz system satisfies the condition $a_{12}a_{21} > 0$, where a_{12} and a_{21} are corresponding elements in the constant matrix $A = (a_{ij})_{3\times 3}$ for the linear part of system [3].

Lorenz system is not integrable and it is difficult to find an analytical solution for this system in three dimension parameters space, but special cases for Lorenz system are studied before studying periodic solutions, and Lorenz studied the system when $\sigma = 10$, $\beta = 8/3$. From the definition of equilibrium points, it is easy to verify that, when $r \leq 1$ the Lorenz system has only one equilibrium point, which is the origin, but when $r > 1$ it has three equilibria points: $E_1(0,0,0)$ $E_{2,3}(\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}\ , r-1)$ [4,5], The Lorenz system has some simple properties such that this system has natural symmetry $(x,y,z) \to (-x,-y,z)$ and the z-axis is invariant [6,7].

[8,9] discussed the stability of Lorenz system about the equilibria points, and found the roots of characteristic equation for this system at origin, but at the second equilibrium point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}\ , r-1)$, the Ref [3] used Routh-Hurwitz test to investigated the stability without founding the roots, the Routh-Hurwitz paly important role in stability of dynamical systems

[10], while the Ref [5] depended on the value of r to investigated the stability without founding the roots.

In [9] studied the stability for system derived from the Lorenz system and depended on the roots to determine the stability at origin, The determination of the roots of a cubic equation in general is fairly difficult, but in the given case, one root is easily found [11]. We can find the roots of equations for third degree by numerical method, and these roots are proximal (not exact), but by using Gardano's method on the same equations we can find exact roots, and by these roots we can investigated the stability for any system.

In this paper, the stability conditions of Lorenz system at the second equilibrium point E_2 is established by using the general formula Gardano's method to find the roots of the characteristic equation for this system. The Lorenz system is described by:

$$\begin{cases} x = \sigma(y - x) \\ y = rx - y - xz \\ z = xy - \beta z \end{cases} \tag{1}$$

Where σ, r, β are positive parameters. Figure 1 and Figure 2 shows the attractors of the system (1).

The approximating linear system (1) at E_1 is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2}$$

or the characteristic equation of the form:

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{3}$$

Then

$$\begin{cases} a = \sigma + \beta + 1 \\ b = \beta(\sigma + 1) + \sigma(1 - r) \\ c = \beta\sigma(1 - r) \end{cases} \tag{4}$$

The solutions of Eq. 3 are

$$\lambda_{1,2} = \frac{1}{2}\left[-\sigma - 1 \pm \sqrt{(\sigma - 1)^2 + 4\sigma r}\ \right], \quad \lambda_3 = -\beta \tag{5}$$

Now consider the system (1) at second equilibrium point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$, Under the linear transformations $(x, y, z) \to (X, Y, Z)$,

$$\begin{cases} x = X + \sqrt{\beta(r-1)} \\ y = Y + \sqrt{\beta(r-1)} \\ z = Z + (r-1) \end{cases} \tag{6}$$

The system (1) becomes

$$\begin{cases} X = -\sigma X + \sigma Y \\ Y = X - Y - \sqrt{\beta(r-1)}\, Z \\ Z = \sqrt{\beta(r-1)} X + \sqrt{\beta(r-1)}\, Y - \beta Z \end{cases} \tag{7}$$

The approximating linear system (1) at equilibrium point E_2 is:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(r-1)} \\ \sqrt{\beta(r-1)} & \sqrt{\beta(r-1)} & -\beta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{8}$$

And the characteristic equation of the form:

$$\lambda^3 + a_1\lambda^2 + b_1\lambda + c_1 = 0 \tag{9}$$

Then

$$\begin{cases} a = a_1 = \sigma + \beta + 1 \\ b_1 = \beta(\sigma + r) \\ c_1 = 2\beta\sigma(r - 1) \end{cases} \tag{10}$$

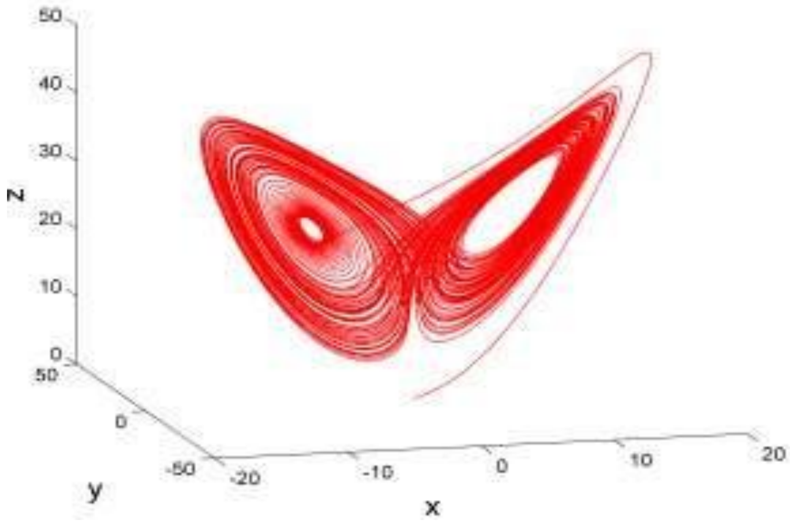


Figure 1 The attractor of system (1) in x, y, z space.

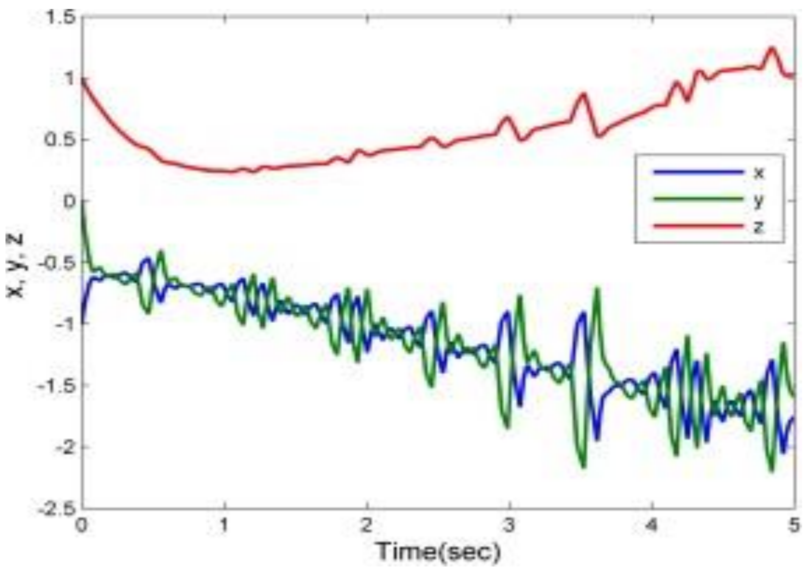


Figure 2 The attractor of system (1).

2. Helping results

Remark 1[5]:

When $\sigma = 10, \beta = 8/3$, the solutions of equation (9) depend on the parameter r as follows:

1- For $1 < r < r_1 \cong 1.3456$, there are three negative real roots,

2- For $r_1 < r < r_2 \cong 24.737$, there are one negative real root and two complex roots with negative real parts,

3- For $r > r_2$ there are one negative real root and two complex roots with positive real parts.

Remark 2[4]:

Let A be a $n \times n$ matrix of constants. A equilibrium point for the system $\mathrm{X} = Ax$ is

- aymptotically stable if all roots of A has negative real parts
- unstable if A has at least one root with a positive real parts.

Remark 3[10]: **Critical case**

In critical cases when the real parts of all roots of the characteristic equation are non positive, with the real part of at least one root being zero.

Remark 4[3]: Critical value

Lorenz system has critical value which is $r_c = 1$ at origin and $r_c = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}$ at the second equilibrium point, and this system is asymptotically stable if r lies between 1 and a critical value r_c at $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$.

Let us denote

$$q = c_1 - \frac{1}{3} a_1 b_1 + \frac{2}{27} a_1^3 \qquad (11)$$

$$\Delta = c_1^2 + \frac{4}{27} b_1^3 - \frac{2}{3} a_1 b_1 c_1 - \frac{1}{27} a_1^2 b_1^2 + \frac{4}{27} a_1^3 c_1 \qquad (12)$$

We will use the following theorem, which enables us to find the exact roots for cubic equation (three degree).

Theorem 1[12, 13]: (Gardano's method)

- If $\Delta = 0$, then the second term of equation (9) has three roots, but one is multiple:

$$\lambda_1 = -2\sqrt[3]{\frac{q}{2}} - \frac{a_1}{3} \ , \quad \lambda_{2,3} = \sqrt[3]{\frac{q}{2}} - \frac{a_1}{3} \qquad (13)$$

- If $\Delta < 0$, then the equation (10) has three different real roots as:

$$\lambda_{i+1} = \sqrt[6]{16(q^2 - \Delta)} \cos \frac{\cos^{-1}\frac{-g}{\sqrt{g^2-\Delta}} + 2\pi i}{3} - \frac{a_1}{3}, \qquad i = 0,1,2. \qquad (14)$$

- If $\Delta > 0$, then the equation (10) has one real root and two complexes conjugate roots with non-vanishing imaginary parts as:

$$\begin{cases} \lambda_1 = \sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}} - \frac{a_1}{3} \\ \lambda_2 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a_1}{3} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) \\ \lambda_3 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a_1}{3} - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) \end{cases} \qquad (15)$$

3. Main results:

Corollary 1: If $\sigma = 10, \beta = 8/3$ and

- $r = r_1$, then we have three negative real roots.
- $r_1 = r_2$, then we have one negative real root and two complex roots with vanishing real parts.

Theorem 2: The solutions of system (1) at second point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$ when $\sigma = 10, \beta = 8/3$ are:

- Asymptotically stable if the following cases hold:
  (i) $\Delta < 0$ and $r \in (1, r_1]$
  (ii) $\Delta > 0$ and $r \in (r_1, r_2)$
- Unstable if $\Delta > 0$ and $r \in (r_2, \infty)$
- Critical case if $\Delta > 0$ and $r = r_2$

Proof:

Case 1: By theorem (1) when $\Delta < 0$ we obtain:
$\lambda_1, \lambda_2, \lambda_3$ are different real roots and these roots are negative when $1 < r \le r_1$, (by Remark 1.1 and corollary1.1), hence satisfied Remark 2.1, therefore the system (1) is asymptotically stable,

When $\Delta > 0$ we obtain:
λ_1 is a real root and λ_2, λ_3 are complex conjugate roots and these roots are negative (negative real parts) when and $r \in (r_1, r_2)$ and satisfied remark 2.1, therefore the system (1) is asymptotically stable.

Case 2: when $\Delta > 0$, we have λ_1 is a real root and λ_2, λ_3 are complex conjugate roots and by remark 1.3.When $r \in (r_2, \infty)$, we obtain that λ_1 is negative and λ_2, λ_3 are positive real part, hence satisfied remark 2,2 , therefore the system (1) is unstable.

Case 3: When $r = r_2$ we have λ_1 is negative real root and $\text{Re}\,\lambda_2 = \text{Re}\,\lambda_3 = 0$ (Corollary1.2) and satisfied remark 3, hence the system (1) is a critical case, the proof is complete.

We can generalization Theorem (2) for any value of σ and β in the following theorem

Theorem 3: The solutions of system (1) at second critical point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$ are:

- Asymptotically stable if the following cases hold:
  (i) $\Delta < 0$, $\lambda_2, \lambda_3 < 0$
  (ii) $\Delta > 0$, $\text{Re}\,\lambda_2 < 0$
- Unstable if $\Delta > 0$, $\text{Re}\,\lambda_2 > 0$

Critical case if $\Delta > 0$, $\text{Re}\,\lambda_2 > 0$

Proof:

Case 1. By theorem (1) when $\Delta < 0$ we obtain: a three different real roots and λ_2, λ_3 are negative (given), we must prove that λ_1 is negative, Since the cubic equation (10) has a positive coefficients therefore, then at least one of these roots is negative real part, hence satisfied remark 2,1 and the system (1) is asymptotically stable,

When $\Delta > 0$, then we have $\text{Re}\,\lambda_3 < 0$ also since $\text{Re}\,\lambda_2 = \text{Re}\,\lambda_3$ (two complex conjugate roots) and λ_1 is negative real root (cubic equation with positive coefficients must have a negative real root), then we have one negative real root and complex roots with negative real part, hence the system (1) is asymptotically stable.

Case 2. By the same theorem when $\Delta > 0$, we have Re $\lambda_3 > 0$ since Re $\lambda_2 =$ Re λ_3 (analogously as in proof of case 1) and λ_1 is negative real root, then we have one negative real root and two complex roots with positive real part, hence satisfied remark 2,2 , the system (1) is unstable.
Case 3. when $\Delta > 0$ and Re $\lambda_2 = 0$, then Re $\lambda_3 = 0$ and λ_1 is negative real part, hence satisfied remark 3, the system (1) is a critical case, the proof is complete.

Proposition:

The necessary and sufficient condition for a second critical point whose parameter σ is greater than the parameter β plus one .

Proof:

1- If $\sigma = \beta$ then critical value become $r_c = -\sigma(\sigma + \beta + 3)$, (contradictions) since r_c must larger then 1.
2- If $\sigma < \beta$ then denominator of critical value is always negative therefore r_c is a negative (contradictions) the same reason of the first case.
3- If $\sigma > \beta$, then critical value is positive and larger then 1(possible), but if $\sigma = \beta + 1$ then critical value become $r_c = \frac{\sigma(\sigma+\beta+3)}{0}$, (contradictions) not allowed dividing on a zero, therefore we must $\sigma > \beta + 1$ only.

4. Comparison

The following Table 1 distinguish the most important differences between the first critical point $E_1(0,0,0)$ and the second point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$.

Table 1 Comparison between equilibria points: E_1, E_2

Nu.	At $E_1(0,0,0)$	Nu.	$E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$,
1-	r is any number larger then zero	1-	r is any number larger then one
2-	A critical value is fixed	2-	A critical value is not fixed, changes with the initial data σ, β
3-	It is easy to determination one root $\lambda = -\beta$	3-	It is difficult to determine one root
4-	Contain only real roots	4-	Contain real and complex roots
5-	Not all coefficients of characteristics equation are positive	5-	all coefficients of characteristics equation are positive
6-	Asymptotically stable when $r \in (0,1)$	6-	Asymptotically stable when $r \in (1, r_c)$
7-	Unstable when $r \in (1, \infty)$	7-	Unstable when $r \in (r_c, \infty)$
8-	Critical case when $r = 1$	8-	Critical case when $r = r_c$
9-	$\sigma = \beta$, $\sigma < \beta$, $\sigma > \beta$	9-	$\sigma > \beta + 1$ only
10 -	q may by negative or positive or zero	10 -	q is a positive only
11-	Δ may by negative or zero	11-	Δ may by negative or positive
12 -	Contain on multiple real roots	12 -	Not contain on multiple real roots

Figures 3, 4 show the stability at equilibria points E_1, E_2 respectively.

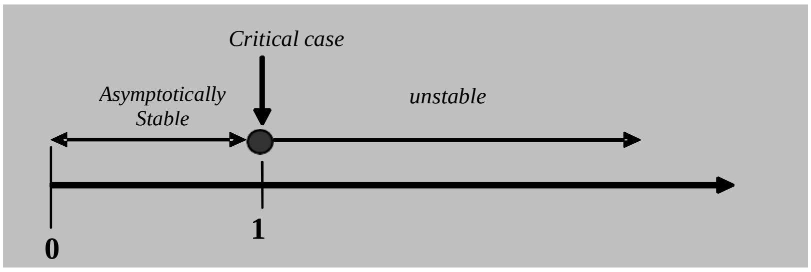


Figure 3 Stability at $E_1(0,0,0)$

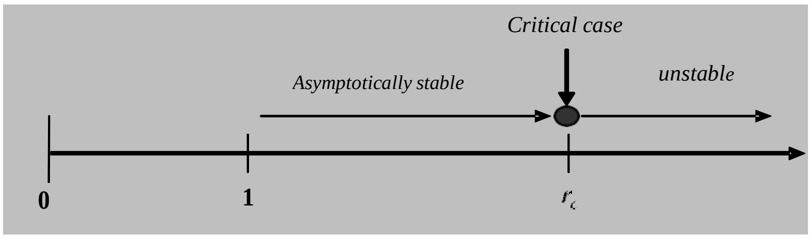


Figure 4 Stability at $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$

5. Illustrative Examples:

In this section, we take two different systems, for example to show how to use the results obtained in this paper to analyze the stability of class chaotic systems.

Example 1: Investigate for stability of the following Lorenz system

$$\begin{cases} x = -10x + 10y \\ y = 11x - y - xz \\ z = xy - \frac{8}{3}z \end{cases}$$

The characteristic equation of Lorenz system is of the form: $\lambda^3 + \frac{41}{3}\lambda^2 + 56\lambda + \frac{1600}{3} = 0$

$q = \frac{22898}{49}$, $\Delta = \frac{436677}{2} > 0$, $11 \in (r_1, r_2)$ and $\lambda_1 = -\frac{1807}{150}$, $\lambda_{2,3} = -\frac{1961}{2421} \pm \frac{1634}{235}i$

Then the system (1) is asymptotically stable

<u>In case r = 100</u>

The characteristic equation of Lorenz system is of the form: $\lambda^3 + \frac{41}{3}\lambda^2 + \frac{880}{3}\lambda + 5280 = 0$

$q = \frac{57859}{14}$, $\Delta = 18907821 > 0$, $100 \in (r_2, \infty)$ and $\lambda_1 = -\frac{8407}{526}$, $\lambda_{2,3} = \frac{564}{487} \pm \frac{2485}{137}i$

Then the system (1) is unstable. Figure 5 and figure 6 show the attractors of system(1) when $\sigma = 10, \beta = 8/3$, (a) $r = 11$, (b) $r = 100$.

Example 2: Investigate for stability of the following Lorenz system

$$\begin{cases} x = -7x + 7y \\ y = \frac{3}{2}x - y - xz \\ z = xy - 4z \end{cases}$$

The characteristic equation of Lorenz system is of the form: $\lambda^3 + 12\lambda^2 + 34\lambda + 28 = 0$

$q = 20$, $\Delta = -\frac{176}{27} < 0$, $\frac{3}{2} \in (1, r_c)$ and $\lambda_1 = -2$, $\lambda_2 = -\frac{2015}{1197}$ $\lambda_3 = -\frac{3152}{379}$

Then the system (1) is asymptotically stable

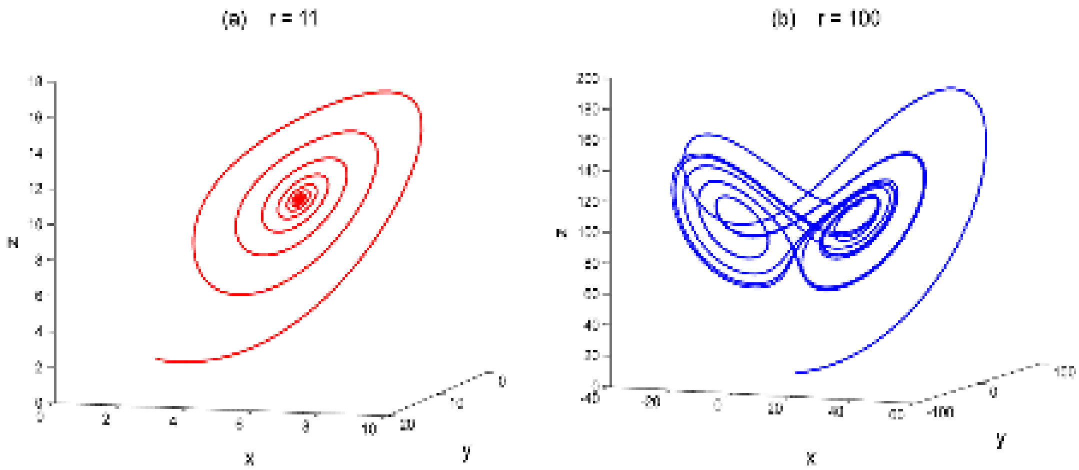


Figure 5 The attractor of system(1) when $\sigma = 10, \beta = 8/3$ (a) $r = 11$, (b) $r = 100$

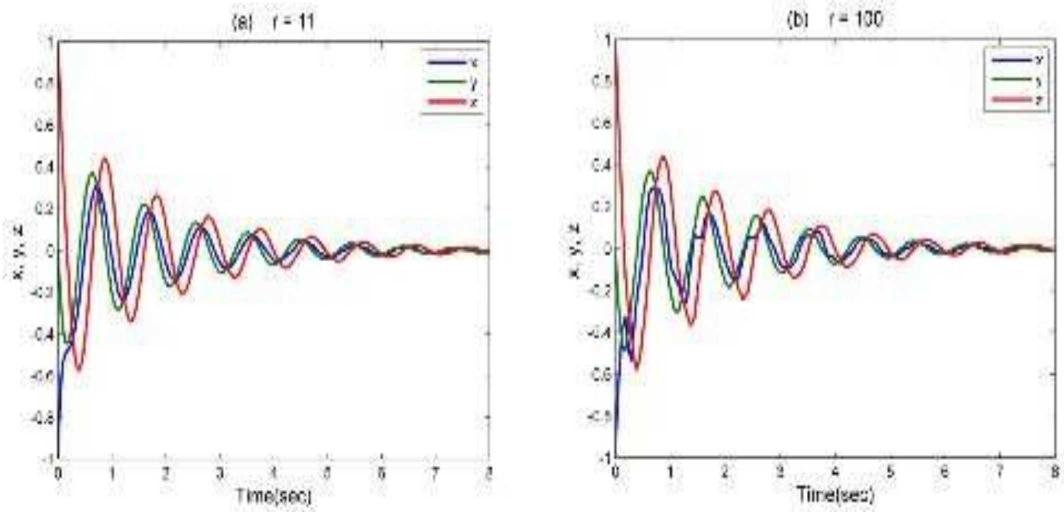


Figure 6 The attractor of system(1) convergent to zero when $\sigma = 10, \beta = 8/3$
(a) $r = 11$, (b) $r = 100$

<u>In case r = 49</u>

The characteristic equation of Lorenz system is of the form: $\lambda^3 + 12\lambda^2 + 224\lambda + 2688 = 0$

$q = 1920$, $\Delta = 4494071 > 0$, $49 = r_c$ and $\lambda_1 = -12$, $\lambda_{2,3} = \pm\frac{13455}{899}i$

Then the system (1) is a critical case. Figure 7 and figure 8 show the attractors of system(1) when $\sigma = 7, \beta = 4$, (a) $r = 3/2$, (b) $r = 49$.

6. Conclusions

In this paper, we have investigated the stability of Lorenz system at the second critical point by using a new method. By this method we justified the same results which found by previous methods. An illustrative examples show the effectiveness and feasibility of this method.

References

[1] Li D, Lu J A, Wu X and Chen G 2005 *Chaos, Solitons & Fractals* **23** 529-534
[2] Al-Obeidi A S and AL-Azzawi S F 2019 *Indonesian Journal of Electrical Engineering and Computer Science* **16** 692-700
[3] Al-Azzawi S F 2012 *Applied Mathematics and Computation* **219** 1144-1152
[4] Borrelli R L and Coleman C S 1998 *Differential Equations* (New York: John Wiley and Sons)
[5] Boyce W E and Diprima R C 2004 *Elementary Differential Equations and Boundary Value Problems* (New York: John Wiley and Sons, Inc)
[6] Zhang, Zeng Y and Li Z 2018 *Chinese Journal of Physics* **56** 793-806
[7] Sambas A, Vaidyanathan S, Zhang S, Zeng Y, Mohamed M A and Mamat M 2019 *IEEE Access* **7** 115454-115462
[8] Lynch S 2001 *Dynamical Systems with Applications using Maple* (Basel: Birkhauser)
[9] Tigan G 2004 *Proceedings of the Third International Colloquium on Mathematics in Engineering and Numerical Physics* 265-272
[10] AL-Azzawi S F and Aziz M M 2019 *Telkomnika* **17** 1931-1940
[11] Ambrosio L and Dancer N 2012 *Calculus of variations and partial differential equations: topics on geometrical evolution problems and degree theory* (German: Springer Science & Business Media)
[12] Aziz M M and Al-Azzawi S F 2017 *Optik* **138** 328-340
[13] Al-Azzawi S F and Aziz M M 2018 *Alexandria engineering journal* **57** 3493-3500

Journal of Physics: Conference Series

PAPER • OPEN ACCESS

Stability of Lorenz System at the Second Equilibria Point based on Gardano's Method

To cite this article: S. F. AL-Azzawi *et al* 2020 *J. Phys.: Conf. Ser.* **1477** 022009

View the article online for updates and enhancements.

Stability of Lorenz System at the Second Equilibria Point based on Gardano's Method

S. F. AL-Azzawi[1,*], Mujiarto[2], L. Patria[3], A. Sambas[2] and W. S. M. Sanjaya[4]

[1] Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq
[2] Department of Mechanical Engineering, Universitas Muhammadiyah Tasikmalaya, Indonesia
[3] Department of Information Systems, Universitas Terbuka, Indonesia
[4] Department of Physics, Universitas Islam Negeri Sunan Gunung Djati, Bandung, Indonesia

*saad_fawzi78@yahoo.com

Abstract. In this paper, stability conditions of the Lorenz system at the second equilibrium point are investigated by applying Gardano's method where the system has three equilibria points. Most of the previous work focused their studies at the original point. A few studies demonstrated stability of dynamical systems at another equilibria points by use of conventional techniques. However, it is often unclear and based on numerical methods. This reason, motivate us to establish the stability conditions of the Lorenz system at a point which different from the origin point and compare between them. Finally, An illustrative example shows the effectiveness and feasibility of this method.

1. Introduction

In 1963, Lorenz found the first chaotic system, which is a third order autonomous system with only two multiplication-type quadratic terms, but displays very complex dynamical behaviors [1,2]. By definition Vanecek and Celikovsky the Lorenz system satisfies the condition $a_{12}a_{21} > 0$, where a_{12} and a_{21} are corresponding elements in the constant matrix $A = (a_{ij})_{3\times3}$ for the linear part of system [3].

Lorenz system is not integrable and it is difficult to find an analytical solution for this system in three dimension parameters space, but special cases for Lorenz system are studied before studying periodic solutions, and Lorenz studied the system when $\sigma = 10, \beta = 8/3$. From the definition of equilibrium points, it is easy to verify that, when $r \leq 1$ the Lorenz system has only one equilibrium point, which is the origin, but when $r > 1$ it has three equilibria points: $E_1(0,0,0)$ $E_{2,3}(\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1)$ [4,5], The Lorenz system has some simple properties such that this system has natural symmetry (x,y,z)→(-x,-y,z) and the z-axis is invariant [6,7].

[8,9] discussed the stability of Lorenz system about the equilibria points, and found the roots of characteristic equation for this system at origin, but at the second equilibrium point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$, the Ref [3] used Routh-Hurwitz test to investigated the stability without founding the roots, the Routh-Hurwitz paly important role in stability of dynamical systems

[10], while the Ref [5] depended on the value of r to investigated the stability without founding the roots.

In [9] studied the stability for system derived from the Lorenz system and depended on the roots to determine the stability at origin, The determination of the roots of a cubic equation in general is fairly difficult, but in the given case, one root is easily found [11]. We can find the roots of equations for third degree by numerical method, and these roots are proximal (not exact), but by using Gardano's method on the same equations we can find exact roots, and by these roots we can investigated the stability for any system.

In this paper, the stability conditions of Lorenz system at the second equilibrium point E_2 is established by using the general formula Gardano's method to find the roots of the characteristic equation for this system. The Lorenz system is described by:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - \beta z \end{cases} \quad (1)$$

Where σ, r, β are positive parameters. Figure 1 and Figure 2 shows the attractors of the system (1).

The approximating linear system (1) at E_1 is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (2)$$

or the characteristic equation of the form:

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \quad (3)$$

Then

$$\begin{cases} a = \sigma + \beta + 1 \\ b = \beta(\sigma + 1) + \sigma(1 - r) \\ c = \beta\sigma(1 - r) \end{cases} \quad (4)$$

The solutions of Eq. 3 are

$$\lambda_{1,2} = \frac{1}{2}\left[-\sigma - 1 \pm \sqrt{(\sigma - 1)^2 + 4\sigma r}\right], \quad \lambda_3 = -\beta \quad (5)$$

Now consider the system (1) at second equilibrium point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$, Under the linear transformations $(x, y, z) \to (X, Y, Z)$,

$$\begin{cases} x = X + \sqrt{\beta(r-1)} \\ y = Y + \sqrt{\beta(r-1)} \\ z = Z + (r-1) \end{cases} \quad (6)$$

The system (1) becomes

$$\begin{cases} \dot{X} = -\sigma X + \sigma Y \\ \dot{Y} = X - Y - \sqrt{\beta(r-1)}\, Z \\ \dot{Z} = \sqrt{\beta(r-1)} X + \sqrt{\beta(r-1)}\, Y - \beta Z \end{cases} \quad (7)$$

The approximating linear system (1) at equilibrium point E_2 is:

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(r-1)} \\ \sqrt{\beta(r-1)} & \sqrt{\beta(r-1)} & -\beta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (8)$$

And the characteristic equation of the form:

$$\lambda^3 + a_1\lambda^2 + b_1\lambda + c_1 = 0 \tag{9}$$

Then

$$\begin{cases} a = a_1 = \sigma + \beta + 1 \\ b_1 = \beta(\sigma + r) \\ c_1 = 2\beta\sigma(r-1) \end{cases} \tag{10}$$

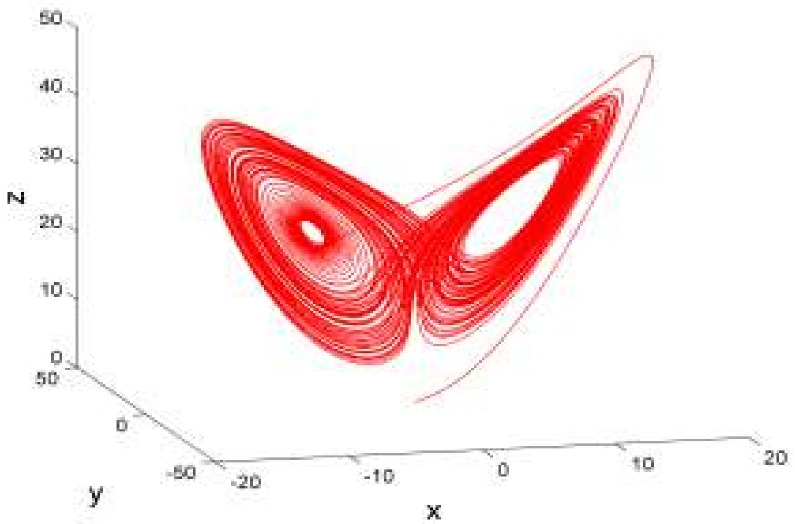


Figure 1 The attractor of system (1) in x, y, z space.

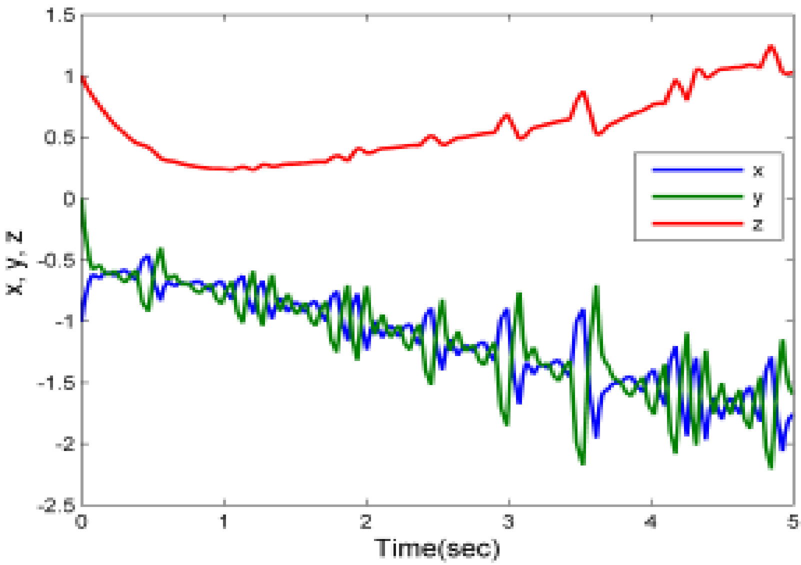


Figure 2 The attractor of system (1).

2. Helping results

Remark 1[5]:

When $\sigma = 10, \beta = 8/3$, the solutions of equation (9) depend on the parameter r as follows:

1- For $1 < r < r_1 \cong 1.3456$, there are three negative real roots,

2- For $r_1 < r < r_2 \cong 24.737$, there are one negative real root and two complex roots with negative real parts,

3- For $r > r_2$ there are one negative real root and two complex roots with positive real parts.

Remark 2[4]:

Let A be a $n \times n$ matrix of constants. A equilibrium point for the system $\dot{X} = Ax$ is

- aymptotically stable if all roots of A has negative real parts
- unstable if A has at least one root with a positive real parts.

Remark 3[10]: **Critical case**

In critical cases when the real parts of all roots of the characteristic equation are non positive, with the real part of at least one root being zero.

Remark 4[3]: Critical value

Lorenz system has critical value which is $r_c = 1$ at origin and $r_c = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}$ at the second equilibrium point, and this system is asymptotically stable if r lies between 1 and a critical value r_c at $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$.

Let us denote

$$q = c_1 - \frac{1}{3}a_1 b_1 + \frac{2}{27}a_1^3 \qquad (11)$$

$$\Delta = c_1^2 + \frac{4}{27}b_1^3 - \frac{2}{3}a_1 b_1 c_1 - \frac{1}{27}a_1^2 b_1^2 + \frac{4}{27}a_1^3 c_1 \qquad (12)$$

We will use the following theorem, which enables us to find the exact roots for cubic equation (three degree).

Theorem 1[12, 13]: (Gardano's method)

- If $\Delta = 0$, then the second term of equation (9) has three roots, but one is multiple:

$$\lambda_1 = -2\sqrt[3]{\frac{q}{2}} - \frac{a_1}{3} \ , \quad \lambda_{2,3} = \sqrt[3]{\frac{q}{2}} - \frac{a_1}{3} \qquad (13)$$

- If $\Delta < 0$, then the equation (10) has three different real roots as:

$$\lambda_{i+1} = \sqrt[6]{16(q^2 - \Delta)} \ cos\frac{cos^{-1}\frac{-q}{\sqrt{q^2-\Delta}} + 2\pi i}{3} - \frac{a_1}{3}, \qquad i = 0,1,2. \qquad (14)$$

- If $\Delta > 0$, then the equation (10) has one real root and two complexes conjugate roots with non-vanishing imaginary parts as:

$$\begin{cases} \lambda_1 = \sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}} - \frac{a_1}{3} \\ \lambda_2 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a_1}{3} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) \\ \lambda_3 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a_1}{3} - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) \end{cases} \qquad (15)$$

3. Main results:

Corollary 1: If $\sigma = 10, \beta = 8/3$ and

- $r = r_1$, then we have three negative real roots.
- $r_1 = r_2$, then we have one negative real root and two complex roots with vanishing real parts.

Theorem 2: The solutions of system (1) at second point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$ when $\sigma = 10, \beta = 8/3$ are:

- Asymptotically stable if the following cases hold:
  (i) $\Delta < 0$ and $r \in (1, r_1]$
  (ii) $\Delta > 0$ and $r \in (r_1, r_2)$
- Unstable if $\Delta > 0$ and $r \in (r_2, \infty)$
- Critical case if $\Delta > 0$ and $r = r_2$

Proof:

Case 1: By theorem (1) when $\Delta < 0$ we obtain:
$\lambda_1, \lambda_2, \lambda_3$ are different real roots and these roots are negative when $1 < r \leq r_1$, (by Remark 1.1 and corollary1.1), hence satisfied Remark 2.1, therefore the system (1) is asymptotically stable,

When $\Delta > 0$ we obtain:
λ_1 is a real root and λ_2, λ_3 are complex conjugate roots and these roots are negative (negative real parts) when and $r \in (r_1, r_2)$ and satisfied remark 2.1, therefore the system (1) is asymptotically stable.

Case 2: when $\Delta > 0$, we have λ_1 is a real root and λ_2, λ_3 are complex conjugate roots and by remark 1.3.When $r \in (r_2, \infty)$, we obtain that λ_1 is negative and λ_2, λ_3 are positive real part, hence satisfied remark 2,2 , therefore the system (1) is unstable.

Case 3: When $r = r_2$ we have λ_1 is negative real root and $\mathrm{Re}\,\lambda_2 = \mathrm{Re}\,\lambda_3 = 0$ (Corollary1.2) and satisfied remark 3, hence the system (1) is a critical case, the proof is complete.

We can generalization Theorem (2) for any value of σ and β in the following theorem

Theorem 3: The solutions of system (1) at second critical point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$ are:

- Asymptotically stable if the following cases hold:
  (i) $\Delta < 0$, $\lambda_2, \lambda_3 < 0$
  (ii) $\Delta > 0$, $\mathrm{Re}\,\lambda_2 < 0$
- Unstable if $\Delta > 0$, $\mathrm{Re}\,\lambda_2 > 0$

Critical case if $\Delta > 0$, $\mathrm{Re}\,\lambda_2 > 0$

Proof:

Case 1. By theorem (1) when $\Delta < 0$ we obtain: a three different real roots and λ_2, λ_3 are negative (given), we must prove that λ_1 is negative, Since the cubic equation (10) has a positive coefficients therefore, then at least one of these roots is negative real part, hence satisfied remark 2,1 and the system (1) is asymptotically stable,

When $\Delta > 0$, then we have $\mathrm{Re}\,\lambda_3 < 0$ also since $\mathrm{Re}\,\lambda_2 = \mathrm{Re}\,\lambda_3$ (two complex conjugate roots) and λ_1 is negative real root (cubic equation with positive coefficients must have a negative real root), then we have one negative real root and complex roots with negative real part, hence the system (1) is asymptotically stable.

Case 2. By the same theorem when $\Delta > 0$, we have $\text{Re}\,\lambda_3 > 0$ since $\text{Re}\,\lambda_2 = \text{Re}\,\lambda_3$ (analogously as in proof of case 1) and λ_1 is negative real root, then we have one negative real root and two complex roots with positive real part, hence satisfied remark 2,2 , the system (1) is unstable.
Case 3. when $\Delta > 0$ and $\text{Re}\,\lambda_2 = 0$, then $\text{Re}\,\lambda_3 = 0$ and λ_1 is negative real part, hence satisfied remark 3, the system (1) is a critical case, the proof is complete.

Proposition:

The necessary and sufficient condition for a second critical point whose parameter σ is greater than the parameter β plus one .

Proof:

1- If $\sigma = \beta$ then critical value become $r_c = -\sigma(\sigma + \beta + 3)$, (contradictions) since r_c must larger then 1.
2- If $\sigma < \beta$ then denominator of critical value is always negative therefore r_c is a negative (contradictions) the same reason of the first case.
3- If $\sigma > \beta$, then critical value is positive and larger then 1(possible), but if $\sigma = \beta + 1$ then critical value become $r_c = \frac{\sigma(\sigma+\beta+3)}{0}$, (contradictions) not allowed dividing on a zero, therefore we must $\sigma > \beta + 1$ only.

4. Comparison

The following Table 1 distinguish the most important differences between the first critical point $E_1(0,0,0)$ and the second point $E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$.

Table 1 Comparison between equilibria points: E_1, E_2

Nu.	At $E_1(0,0,0)$	**Nu.**	$E_2(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1)$,
1-	r is any number larger then zero	1-	r is any number larger then one
2-	A critical value is fixed	2-	A critical value is not fixed, changes with the initial data σ, β
3-	It is easy to determination one root $\lambda = -\beta$	3-	It is difficult to determine one root
4-	Contain only real roots	4-	Contain real and complex roots
5-	Not all coefficients of characteristics equation are positive	5-	all coefficients of characteristics equation are positive
6-	Asymptotically stable when $r \in (0,1)$	6-	Asymptotically stable when $r \in (1, r_c)$
7-	Unstable when $r \in (1, \infty)$	7-	Unstable when $r \in (r_c, \infty)$
8-	Critical case when $r = 1$	8-	Critical case when $r = r_c$
9-	$\sigma = \beta$, $\sigma < \beta$, $\sigma > \beta$	9-	$\sigma > \beta + 1$ only
10 -	q may by negative or positive or zero	10 -	q is a positive only
11-	Δ may by negative or zero	11-	Δ may by negative or positive
12 -	Contain on multiple real roots	12 -	Not contain on multiple real roots

Figures 3, 4 show the stability at equilibria points E_1, E_2 respectively.

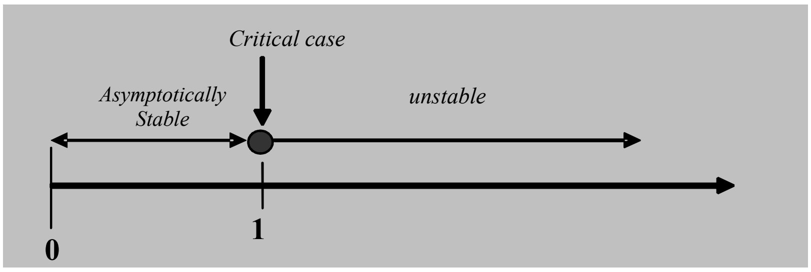


Figure 3 Stability at $E_1(0,0,0)$

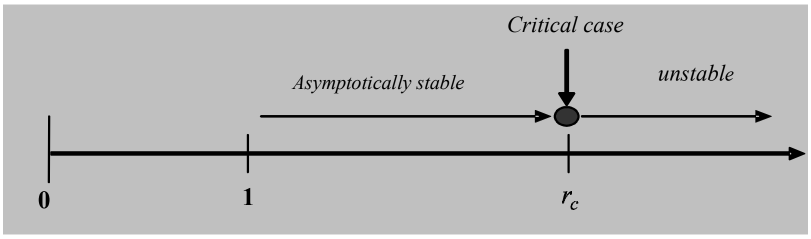


Figure 4 Stability at $E_2(\sqrt{\beta(r-1)},\sqrt{\beta(r-1)},r-1)$

5. Illustrative Examples:

In this section, we take two different systems, for example to show how to use the results obtained in this paper to analyze the stability of class chaotic systems.

Example 1: Investigate for stability of the following Lorenz system

$$\begin{cases} \dot{x} = -10x + 10y \\ \dot{y} = 11x - y - xz \\ \dot{z} = xy - \frac{8}{3}z \end{cases}$$

The characteristic equation of Lorenz system is of the form: $\lambda^3 + \frac{41}{3}\lambda^2 + 56\lambda + \frac{1600}{3} = 0$

$q = \frac{22898}{49}$, $\Delta = \frac{436677}{2} > 0$, $11 \in (r_1, r_2)$ and $\lambda_1 = -\frac{1807}{150}$, $\lambda_{2,3} = -\frac{1961}{2421} \pm \frac{1634}{235}i$

Then the system (1) is asymptotically stable

__**In case r = 100**__

The characteristic equation of Lorenz system is of the form: $\lambda^3 + \frac{41}{3}\lambda^2 + \frac{880}{3}\lambda + 5280 = 0$

$q = \frac{57859}{14}$, $\Delta = 18907821 > 0$, $100 \in (r_2, \infty)$ and $\lambda_1 = -\frac{8407}{526}$, $\lambda_{2,3} = \frac{564}{487} \pm \frac{2485}{137}i$

Then the system (1) is unstable. Figure 5 and figure 6 show the attractors of system(1) when $\sigma = 10, \beta = 8/3$, (a) $r = 11$, (b) $r = 100$.

Example 2: Investigate for stability of the following Lorenz system

$$\begin{cases} \dot{x} = -7x + 7y \\ \dot{y} = \frac{3}{2}x - y - xz \\ \dot{z} = xy - 4z \end{cases}$$

The characteristic equation of Lorenz system is of the form: $\lambda^3 + 12\lambda^2 + 34\lambda + 28 = 0$

$q = 20$, $\Delta = -\frac{176}{27} < 0$, $\frac{3}{2} \in (1, r_c)$ and $\lambda_1 = -2$, $\lambda_2 = -\frac{2015}{1197}$ $\lambda_3 = -\frac{3152}{379}$

Then the system (1) is asymptotically stable

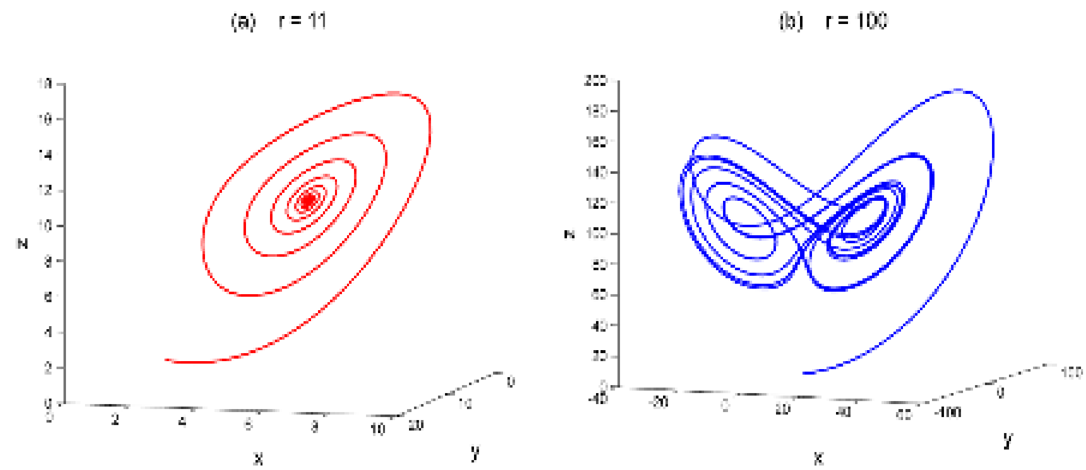


Figure 5 The attractor of system(1) when $\sigma = 10, \beta = 8/3$ (a) $r = 11$, (b) $r = 100$

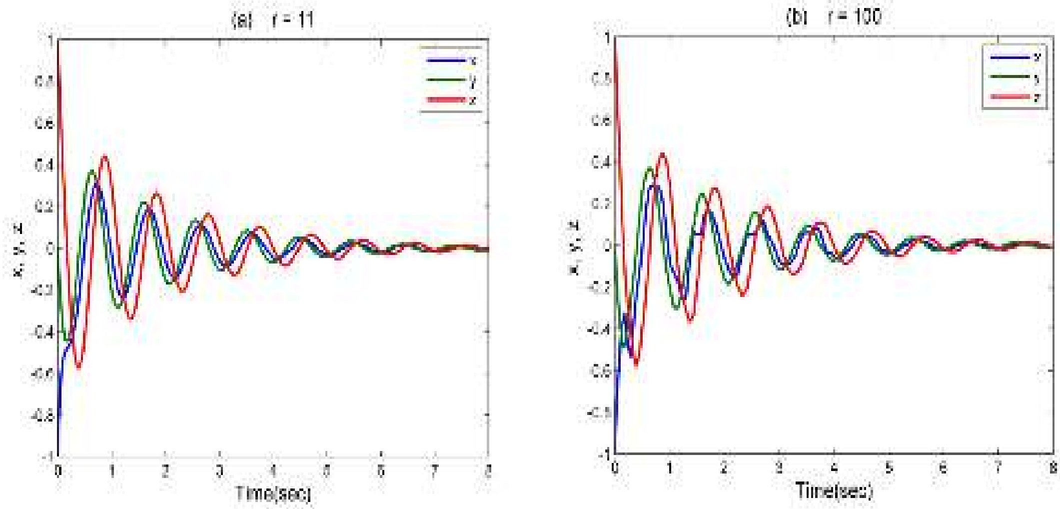


Figure 6 The attractor of system(1) convergent to zero when $\sigma = 10, \beta = 8/3$
(a) $r = 11$, (b) $r = 100$

<u>In case r = 49</u>

The characteristic equation of Lorenz system is of the form: $\lambda^3 + 12\lambda^2 + 224\lambda + 2688 = 0$

$q = 1920$, $\Delta = 4494071 > 0$, $49 = r_c$ and $\lambda_1 = -12$, $\lambda_{2,3} = \pm\frac{13455}{899}i$

Then the system (1) is a critical case. Figure 7 and figure 8 show the attractors of system(1) when $\sigma = 7, \beta = 4$, (a) $r = 3/2$, (b) $r = 49$.

6. Conclusions

In this paper, we have investigated the stability of Lorenz system at the second critical point by using a new method. By this method we justified the same results which found by previous methods. An illustrative examples show the effectiveness and feasibility of this method.

References

[1] Li D, Lu J A, Wu X and Chen G 2005 *Chaos, Solitons & Fractals* **23** 529-534
[2] Al-Obeidi A S and AL-Azzawi S F 2019 *Indonesian Journal of Electrical Engineering and Computer Science* **16** 692-700
[3] Al-Azzawi S F 2012 *Applied Mathematics and Computation* **219** 1144-1152
[4] Borrelli R L and Coleman C S 1998 *Differential Equations* (New York: John Wiley and Sons)
[5] Boyce W E and Diprima R C 2004 *Elementary Differential Equations and Boundary Value Problems* (New York: John Wiley and Sons, Inc)
[6] Zhang, Zeng Y and Li Z 2018 *Chinese Journal of Physics* **56** 793-806
[7] Sambas A, Vaidyanathan S, Zhang S, Zeng Y, Mohamed M A and Mamat M 2019 *IEEE Access* **7** 115454-115462
[8] Lynch S 2001 *Dynamical Systems with Applications using Maple* (Basel: Birkhauser)
[9] Tigan G 2004 *Proceedings of the Third International Colloquium on Mathematics in Engineering and Numerical Physics* 265-272
[10] AL-Azzawi S F and Aziz M M 2019 *Telkomnika* **17** 1931-1940
[11] Ambrosio L and Dancer N 2012 *Calculus of variations and partial differential equations: topics on geometrical evolution problems and degree theory* (German: Springer Science & Business Media)
[12] Aziz M M and Al-Azzawi S F 2017 *Optik* **138** 328-340
[13] Al-Azzawi S F and Aziz M M 2018 *Alexandria engineering journal* **57** 3493-3500